Profile and scaling of the fractal exponent of percolations in complex networks

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We propose a novel finite size scaling analysis for percolation transition observed in complex networks. While it is known that the cooperative systems in growing network models often undergo an infinite order transition with inverted Berezinskii-Kosterlitz-Thouless singularity, the transition point is very hard for numerical simulations to find because the system is in a critical state outside the ordered phase. We propose a finite size scaling form for the order parameter by using the fractal exponent, which enables us to determine the transition points and critical exponents numerically for *infinite* order transitions as well as standard second order transitions. We confirm the validity of our scaling hypothesis through the Monte-Carlo simulations for bond percolations in two network models; the growing random network model, which exhibits an infinite order transition, and its reconstructed network, which shows a conventional second order transition.

PACS numbers: 89.75.Hc 64.60.aq 89.65.-s

I. INTRODUCTION

The science of complex networks has led to new perspectives in the study of the statistical physics [1–4]. Various processes, such as percolations, epidemic spreadings, spin systems, and coupled oscillators, in complex networks have been actively studied (see [5] and references therein).

An elementary theoretical framework of critical phenomena in complex networks has been provided in the manner of the mean field approximation and *local tree approximation* [5]. Particularly, the equilibrium systems on the random graphs with arbitrary degree distributions $P(k)$ (k denotes degree), so-called configuration model, are well described by local tree approximation, where critical properties are determined only by the power exponent γ of the degree distribution, $P(k) \propto k^{-\gamma}$ [2, 5]. On the other hand, the systems on network models made with growth mechanism are not the case. Let us consider the bond percolation with open bond probability p. Several analytical works for the percolations on growing network models reveal that the system exhibits unusual phase transitions, termed infinite order transition with inverted *Berezinskii-Kosterlitz-Thouless (BKT) singularity* [6–18] (see also [19–21] for the case of spin systems): (i) The singularity of the phase transition is infinitely weak. When p is larger than the transition point p_c, the order parameter $m \equiv S_{\max}(N)/N$, where $S_{\max}(N)$ is the mean size of the largest clusters over percolation trials in the system with N nodes, follows

$$m \propto \exp(-\alpha/(\Delta p)^{\beta'}), \tag{1}$$

where $\Delta p = p - p_c$. (ii) Below the transition point, the mean number n_s of clusters with size s per node obeys the power law,

$$n_s \propto s^{-\tau}. \tag{2}$$

Recent studies [17, 18] revealed that this phase is similar to the critical phase (also termed intermediate phase [22, 23]) observed on (transitive) nonamenable graphs (NAGs). NAGs are defined as infinite graphs with positive Cheeger constant [22, 23]. Typical examples of NAGs are hyperbolic lattices and trees. The bond percolation on a NAG takes three distinct phases; (i) the non-percolating phase ($0 \le p < p_{c1}$) in which only finite size clusters exist, (ii) the critical phase ($p_{c1} \le p \le p_{c2}$) in which there are infinitely many infinite clusters, and (iii) the percolating phase ($p_{c2} < p \le 1$) in which the system has a unique infinite cluster. Here an *infinite cluster* means a cluster whose size is of order $O(N^\alpha)$ ($0 < \alpha \le 1$).

It has been proven that the phase boundaries of amenable graphs including the Euclidean lattices, and NAGs satisfy $0 < p_{c1} = p_{c2} \le 1$ and $0 < p_{c1} < p_{c2} \le 1$, respectively [22, 23]. The scenarios of the phase transitions of the percolations on regular graphs are summarized in Table I. If the above-mentioned unusual phase on growing networks is regarded as the critical phase on

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TABLE I: Expected scenario of phase transitions of bond percolations on various graphs. The number of ends of an infinite graph G is defined as the supremum of the number of infinite connected components in $G\backslash S$, where $G\backslash S$ is the graph obtained from G by removing arbitrary finite subgraph S and the edges incident to those [22, 23] .

graph type	example	scenario
amenable graph with two ends	chain	$0 < p_{c1} = p_{c2} = 1$
amenable graph with one end	$d(\geq 2)$-dimensional Euclidean lattice	$0 < p_{c1} = p_{c2} < 1$
NAG with infinitely many ends	Cayley tree	$0 < p_{c1} < p_{c2} = 1$
NAG with one end	enhanced binary tree [24–26], hyperbolic lattice [27]	$0 < p_{c1} < p_{c2} < 1$
tree under growth mechanism	growing random tree [8, 11, 12, 14, 17]	$0 = p_{c1} < p_{c2} = 1$
network under growth mechanism	growing random network (m-out graph) [11, 28, 29]	
	Callaway-Hopcroft-Kleinberg-Newman-Strogatz model [6, 7]	$0 = p_{c1} < p_{c2} < 1$
hierarchical scale free networks	decorated (2,2)-flower [13, 16, 18], Hanoi network [15]	

NAGs, we may say that the growing networked systems add a new scenario; $p_{c1} = 0$ and $p_{c2} \neq 0$ (last two rows in Table I). However, it is not known which relation (or none of them) holds for complex networked systems, and we should keep the above three phases in mind in studying the critical phenomena. This investigation is a challenging study because numerical studies so far were not conscious of the existence of the critical phase.

Finite scaling analysis is a powerful method to extract the transition point and critical exponents from numerical data. It is indeed the case for some *non-growing* network models as seen in a finite size scaling analysis proposed by Hong et al. [30]. However, to estimate the transition point p_{c2} and critical behaviors of the *growing* networks, we are faced with the following difficulties. The transition at p_{c2} is the one between the percolating phase and the critical phase. We should perform a scaling analysis with data only for $p \geq p_{c2}$ because the phase below p_{c2} is critical so that a standard scaling analysis cannot be applied [17]. Second, the transition at p_{c2} is of infinite order so that conventional critical exponents do not work as normal fitting parameters.

In this paper, we propose a finite size scaling analysis to investigate numerically both the growing and non-growing networked systems. Our finite size scaling form is described in terms of the network size N and the fractal exponent, which characterizes the critical phase [17, 18, 24, 25]. As an application, we evaluate the fractal exponent and perform the scaling analysis for the percolations both on the random attachment growing network [11] (RAGN; a random attachment version of the Barabási-Albert model [31]), which exhibits an infinite order transition with inverted BKT singularities [11, 28, 29], and its reconstructed network (a random network having the degree distribution same as the RAGN), which exhibits a second order transition from the non-percolating phase to the percolating phase. Our finite size scaling analysis works quite well for both cases.

II. SCALING ARGUMENT

To investigate the critical phase, it is useful to profile the fractal exponent ψ [24]. It is defined as

$$\psi \equiv \lim_{N\to\infty} \log_N S_{\max}(N), \tag{3}$$

which mimics d_f/d for d-dimensional Euclidean lattice systems, d_f being the fractal dimension of the largest clusters. In the critical phase, ψ is related to τ in Eq.(2) as $\tau = 1+\psi^{-1}$, indicating that ψ plays a role of a natural cutoff exponent of n_s [17, 18]. Then, the non-percolating phase, the critical phase, and the percolating phase are represented with $\psi = 0$, $0 < \psi < 1$, and $\psi = 1$, respectively.

Let us assume a scaling form with ψ and N in the ordered phase. First, we suppose the existence of crossover size $N^*(p)$ at p. When N is much smaller than the crossover size, $N \ll N^*(p)$, the system behaves as if it were critical even for $p > p_{c2}$; $S_{\max}(N) \propto N^{\psi_c}$, while $S_{\max}(N)$ is proportional to N when $N \gg N^*(p)$. By connecting these two limits, we expect a finite size scaling form for $S_{\max}(N)$ as

$$S_{\max}(N) = N^{\psi_c} f_1\Big(\frac{N}{N^*(p)}\Big), \tag{4}$$

where

$$f_1(x) = \begin{cases} x^{1-\psi_c} & \text{for } x \gg 1 \\ \text{const} & \text{for } x \ll 1, \end{cases} \tag{5}$$

ψ_c being the fractal exponent at $p = p_{c2}$. Or equivalently,

$$S_{\max}(N) = N^*(p)^{\psi_c} f_2\Big(\frac{N}{N^*(p)}\Big), \tag{6}$$

where

$$f_2(x) = \begin{cases} x & \text{for } x \gg 1 \\ x^{\psi_c} & \text{for } x \ll 1. \end{cases} \tag{7}$$

By taking the logarithmic derivative of Eq.(6) with N, we obtain a finite size scaling form for ψ as

$$\psi(N) = \frac{d \ln S_{\max}(N)}{d \ln N} = g\Big(\frac{N}{N^*(p)}\Big), \tag{8}$$

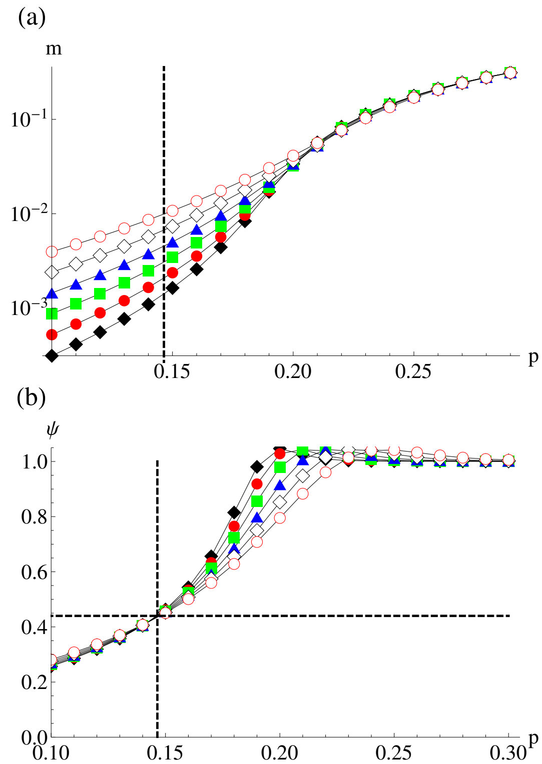

FIG. 1: (Color Online) (a) The order parameter m and (b) the fractal exponent ψ on the RAGN. The numbers of nodes N are 2^{17} (black-diamond), 2^{16} (red-circle), 2^{15} (green-square), 2^{14} (blue-triangle), 2^{13} (open-diamond), and 2^{12} (open-circle). The vertical dashed line indicates $p_{c2} = (1 - 1/\sqrt{2})/2$, and the horizontal dashed line indicates $\psi_c = 0.44$.

where

$$g(x) = \frac{d \ln f_2(x)}{d \ln(x)} = \begin{cases} 1 & \text{for } \ x \gg 1 \\ \psi_c & \text{for } \ x \ll 1. \end{cases} \tag{9}$$

Similarly, the finite size scaling form for the average size $S_{\mathrm{av}}(N) = \sum_{s \neq S_{\max}} s^2 n_s$ of the cluster to which any node belongs can be assumed as

$$S_{\mathrm{av}}(N) = N^{\psi_{\mathrm{av}}} h\Big(\frac{N}{N^*(p)}\Big), \tag{10}$$

where

$$h(x) = \begin{cases} x^{-\psi_{\mathrm{av}}} & \text{for } \ x \gg 1 \\ \text{const} & \text{for } \ x \ll 1, \end{cases} \tag{11}$$

and $\psi_{\mathrm{av}} \equiv 2\psi_c - 1$, which corresponds with the fractal exponent of the mean cluster size at $p = p_{c2}$.

For practical use of Eqs.(4) and (8), we need to know crossover size $N^*(p)$. By considering the case $N \sim N^*(p)$, we relate $N^*(p)$ to $S_{\max}(N)$ and $m(p)$ as

$$S_{\max}(N) \sim N^*(p)^{\psi_c - 1} N \to N^*(p) \sim m(p)^{1/(\psi_c - 1)}. \tag{12}$$

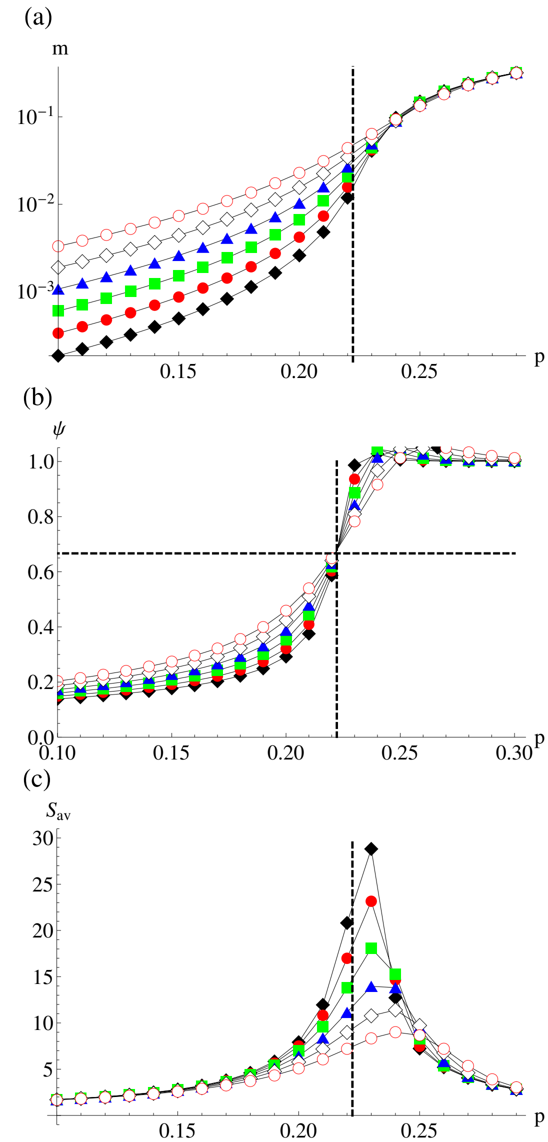

FIG. 2: (a) The order parameter m, (b) the fractal exponent ψ, and (c) the mean cluster size S_{av} on the reconstructed network. The numbers of nodes N are 2^{17} (black-diamond), 2^{16} (red-circle), 2^{15} (green-square), 2^{14} (blue-triangle), 2^{13} (open-diamond), and 2^{12} (open-circle). The vertical dashed line indicates $p_{c2} = 2/9$, and the horizontal dashed line indicates $\psi_c = 2/3$.

In the case of the second order transition, $m(p)$ follows $m \propto (\Delta p)^{\beta}$ for $p \geq p_{c2} = p_{c1}$. Then the finite size scaling form for $S_{\max}$ is

$$S_{\max}(N) = N^{\psi_c} f_1(N(\Delta p)^{\frac{\beta}{1-\psi_c}}), \tag{13}$$

and that for ψ is

$$\psi = g(N(\Delta p)^{\frac{\beta}{1-\psi_c}}). \tag{14}$$

This argument also can be applied to the case of the infinite order transition. Because $m(p)$ follows Eq.(1), we

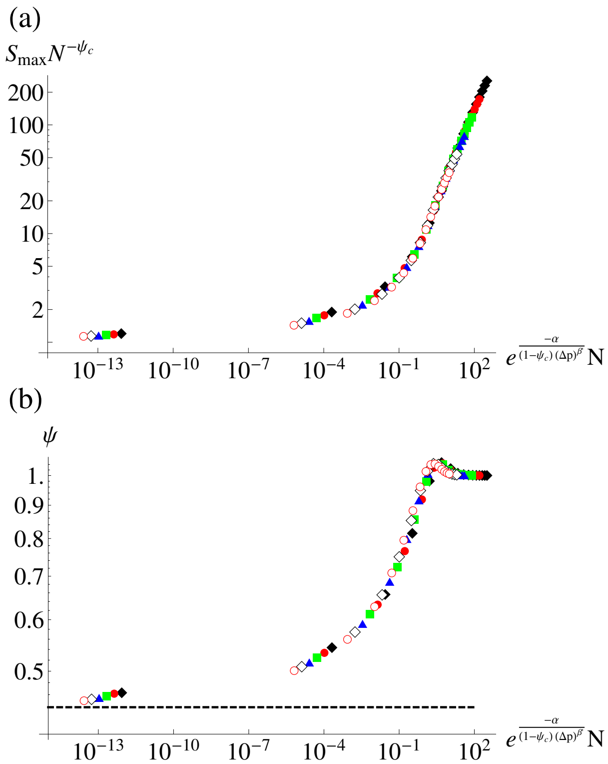

FIG. 3: Finite size scalings for (a) $S_{\max}$ by Eq.(15) and (b) ψ by Eq.(16) on the RAGN. $N = 2^{17}$(black-diamond), 2^{16} (red-circle), 2^{15} (green-square), 2^{14} (blue-triangle), 2^{13} (open-diamond), and 2^{12} (open-circle). The horizontal dashed line follows $\psi_c = 0.44$.

obtain finite size scaling forms of infinite order transition for $S_{\max}(N)$ and ψ by substituting Eq.(12) into Eqs.(4) and (8) as

$$S_{\max}(N) = N^{\psi_c} f_1(Ne^{-\alpha/(1-\psi_c)(\Delta p)^\beta}), \tag{15}$$

and

$$\psi = g(Ne^{-\alpha/(1-\psi_c)(\Delta p)^\beta}), \tag{16}$$

respectively.

III. NUMERICAL CHECK

To check the validity of our scaling argument, we perform the Monte-Carlo simulation for the bond percolations on the RAGN and its reconstructed network. The RAGN stochastically generates a graph realization with N nodes in the following way. We start at time $t = 0$ with a complete graph with $l + 1$ nodes. At each time step, a new node joins into a graph, and links to l randomly-chosen nodes among pre-existing nodes. The network growth continues until $t = N - l - 1$. The stationary degree distribution $P(k)$ of the resulting network is $P(k) \propto (l/(l+1))^k$. A reconstructed network of the

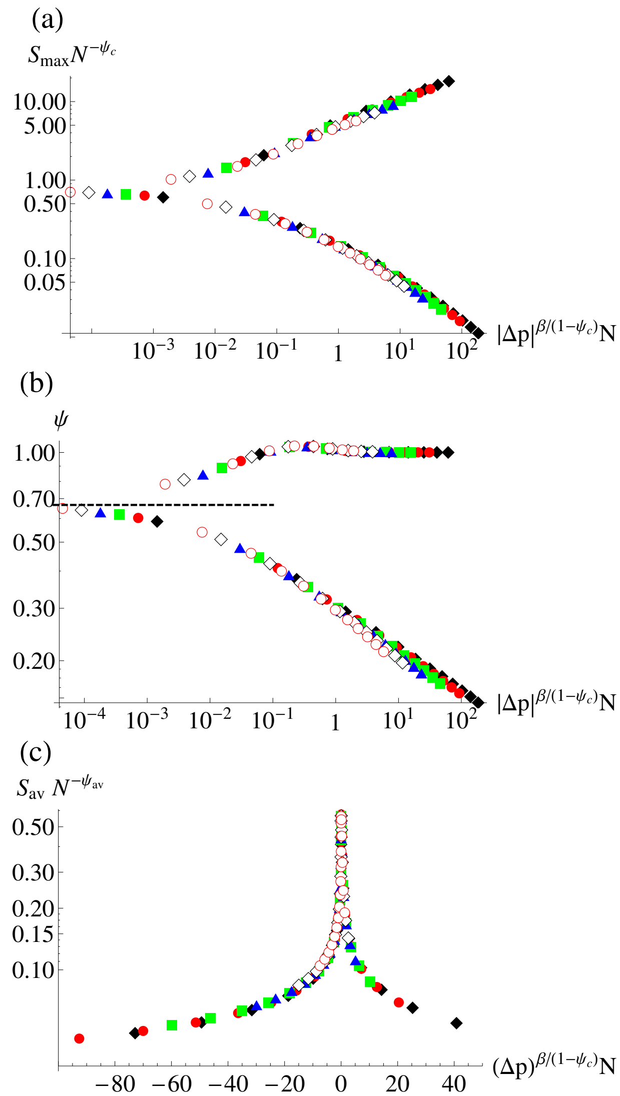

FIG. 4: Finite size scalings for (a) $S_{\max}$ by Eq.(13), (b) ψ by Eq.(14), and Sav by Eq.(10) on the reconstructed network. $N = 2^{17}$(black-diamond), 2^{16} (red-circle), 2^{15} (green-square), 2^{14} (blue-triangle), 2^{13} (open-diamond), and 2^{12} (open-circle). The horizontal dashed line follows $\psi_c = 2/3$.

RAGN is then obtained by reconnecting its edges at random, keeping the degree of each node. This operation generates an uncorrelated network with the same degree distributions as the original one.

The rigorous results of the critical behavior for the bond percolation on the RAGN were given by Bollobás and Riordan [28, 29] ; the transition point $p_{c2}(l)$, above which a giant component having size of $O(N)$ emerges, is given by $p_{c2}(l) = (1 - \sqrt{l-1/l})/l$ and for $p > p_{c2}$ the order parameter follows Eq.(1) with $\alpha = \pi/2[l(l-1)]^{1/4}$ and $\beta' = 1/2$. The bond percolation on the reconstructed network is analyzed by local tree approximation [2, 5]. The transition point is given by $p_c = \langle k\rangle/\langle k^2 - k\rangle = 2/9$. Also, it is known that the percolations on uncorrelated

networks with $P(k) \propto k^{-\gamma}$, where $\gamma > 4$, belong to the mean field universality class, $\beta = 1$.

We set $l = 2$, and $N = 2^{11}, 2^{12}, \cdots, 2^{18}$. The number of graph realizations is 100000 (10000 for $N = 2^{16}, 2^{17}, 2^{18}$), and the number of percolation trials on each realization is 100. The order parameter m and the fractal exponent ψ on the RAGN are shown in Fig. 1. Here we calculate $\psi(N)$ by the difference $[\ln S_{\max}(2N) - \ln S_{\max}(N/2))]/[\ln(2N) - \ln(N/2)]$. As N increases, $\psi(N)$ converges to a certain value depending on p for $p < p_{c2}$, while $\psi(N)$ for $p > p_{c2}$ approaches unity very slowly with N. In the thermodynamic limit, $N \to \infty$, ψ grows continuously with p for $0(= p_{c1}) < p < p_{c2}$ up to $\psi_c \sim 0.44$ at $p = p_{c2} = (1 - \sqrt{1/2})/2$, and then jump to $\psi = 1$. The same behavior of ψ was analytically observed on the decorated (2,2)-flower [18].

The fractal exponent is useful not only to confirm the existence of the critical *phase*, but also to determine the critical *point* of ordinary continuous phase transition. In Fig. 2, m, ψ, and $S_{\rm av}$ on the reconstructed networks are plotted with respect to p. In this case, ψ with several sizes cross at $(p_c, \psi_c) \simeq (2/9, 2/3)$. In the limit $N \to \infty$, $\psi = 0(= 1)$ for $p < p_c(> p_c)$, which means the transition at p_c is the one between the non-percolating phase and the percolating phase. The result that a giant component at the transition point is of $O(N^{2/3})$ is also observed on the Erdös-Rényi model [32].

The finite size scaling results for the RAGN and the reconstructed network are shown in Figs. 3 and 4, respectively. Here fitting parameter β (and also α for the RAGN) is set to the analytically obtained values. We observe good data collapses for both cases, confirming the validity of our scaling hypothesis. We did not perform the finite size scaling of $S_{\rm av}$ for the RAGN. The reason is that our numerical result for the RAGN shows $\psi_c < 1/2$ at the transition point $p = p_{c2}$, leading to $\psi_{\rm av} < 0$. This indicates that the mean cluster size of the RAGN does not diverge, and shows a finite jump at $p = p_{c2}$. This is actually observed on the RAGN [11].

IV. SUMMARY

To summarize, we have proposed a novel scaling analysis for critical phenomena. We have confirmed the validity of our scaling via two exactly-solved models; the RAGN and its reconstructed network.

Note that the present scaling form includes the conventional finite size scaling. For d-dimensional lattice systems, the fractal exponent satisfies $\psi_c = 1 - \beta/d\nu$, so that Eq.(4) reduces to the conventional scaling for Nm provided that $N/N^*(p) = (L/\xi)^d$, where L is the linear dimension, and ν the critical exponent of correlation *length* ξ. In this sense, $N^*(p)$ is regarded as the correlation *volume* [5].

Whether percolations on the other complex networks have such a critical phase or not is not known. Also, whether other dynamics, e.g., epidemic spreading, coupled oscillators and so on, have such a critical phase or not is an interesting open problem. By answering to these questions, the relation between network topology and the dynamics thereon will become more clear. We believe that the present method will contribute to the further analysis.

Acknowledgments

The authors thank M. Sato, K. Konno, and N. Masuda for valuable comments.

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